# CONSTRUCTION OF BALANCED $n$-ARY BLOCK DESIGNS AND PARTIALLY BALANCED ARRAYS

BY

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## 1. INTRODUCTION

Balanced $n$-ary designs were introduced by Tocher (1952) as a generalisation of the BIB designs. For such designs, $V$ treatments are arranged in $B$ blocks each of size $K$ such that every treatment is replicated $R$ times and the sum of products $\sum_{i=1}^{B} n_{ij}\, n_{im}, j \neq m$ is constant ($=\pi$ say) where $n_{ij}$ is the number of times the $j^{th}$ treatment occurs in the $i^{th}$ block $i=1, 1..., B; j=1, \ldots V$ and $n_{ij}$ can take $n$ different positive integral values including zero. In the present paper we have shown that the balance ternary designs can be constructed from the incidence matrix of any BIB design. The ternary designs of Dey (1970) turn out to be the particular cases of these ternary designs. We have also shown that from the incidence matrix of any balanced $t$-ary design, we can always get a balanced $(2t-1)$-ary design. We have also proposed a method through which balanced $n$-ary designs for any $n$ can be constructed by using $(n-1)$-ary design.

## 2. BALANCED $n$-ARY DESIGNS THROUGH BIB DESIGNS

### 2.1 *Method 1*

Consider a BIB design with parameters $(v, k, r, b, \lambda)$. Then the incidence matrix $N_2$ of this binary design is a $bxv$ matrix with elements 0 and 1 where 0 indicates the absence of any treatment and presence of any treatment is indicated by 1. In general we shall represent by $N_t$ the incidence matrix of any $t$-ary design. Consider now another BIB design with parameters $(v^*=v, k^*, r^*, b^*, \lambda^*)$ so that $N^*_2$ is the incidence matrix of the design. We now state the following theorem :

**Theorem 2.1**

A balanced ternary design, in general, can be obtained by adding the elements of pairs $(i, j)$ of rows where the rows $i$ and $j$ are chosen from different incidence matrices $N_2$ and $N^*_2$ of different BIB designs $(v, k, r, b, \lambda)$ and $(v^*=v, k^*, r^*, b, \lambda^*)$. The parameters of the ternary design will be $V=v$, $K=(k+k^*)$, $R=2rr^*+r^*(b-r)+r(b^*-r^*)$, $B=bb^*$, $\pi = 4\lambda\lambda^*+\lambda(b\lambda^*-2r^*+\lambda^*)+4\lambda(r^*-\lambda^*)+(b-2r+\lambda)\lambda^*+4(r-\lambda)\lambda^*+2(r-\lambda)(r^*-\lambda^*)$.

**Proof**

The parameters $V, K, B$ need no proof. Since both the incidence matrices $N_2$ and $N^*_2$ have the elements 0 and 1, it is seen that the element 1 of $N_2$ in any column vector can be added with the element of 1 of $N^*_2$ in $rr^*$ ways and all such additions total to $2rr^*$. Similarly, the element 1 of $N_2$ when added with 0 of $N^*_2$ gives the total $r(b^*-r^*)$ and the element 0 of $N^*_2$ when added with 1 of $N_2$ gives the total $r^*(b-r)$. Thus $R$ comes out to be

$$[2rr^*+r^*(b-r)+r(b^*-r^*)]$$

We now show the constancy of $\pi$. In any two columned submatrix of the incidence matrix $N_2$ the pair of the frequencies (11) occurs $\lambda$ times, the pair (00) occurs $(b-2r+\lambda)$ times and the pairs (01) and (10) occur $2(r-\lambda)$ times. Similarly these pairs occur respectively $\lambda^*$, $(b^*-2r^*+\lambda^*)$ and $2(r^*-\lambda^*)$ times $N^*_2$. When the elements of rows $(i, j)$ of $N_2$ and and $N^*_2$ are added, then the pair (11) when added with the similar pair (11) leads to the pair (22) which has the product 4. Since (11) of $N_2$ can be added with (11) of $N^*_2$ in $\lambda\lambda^*$ ways we have the sum of such products as $4\lambda\lambda^*$. It can similarly be seen that (11) of $N_2$ can be added to (00) of $N_2$ to yield the sum of products

$$\lambda(b-2r^*+\lambda^*),$$

(11) of $N_2$ can be added to (01) and (10) of $N^*_2$ to give the sum of products $4\lambda(r^*-\lambda\ )$,

Pair (00) of $N_2$ when added with the pair (11) of $N^*_2$ gives the sum of products $\lambda^*(b-2r^*+\lambda)$,

Pairs (01) and (10) of $N_2$ when added with the pair (11) of $N^*_2$ yield the sum of products $4(r-\lambda)\lambda^*$, and pairs (01) and (10) of $N_2$ when added with pairs (01) and (10) of $N^*_2$ give the sum of products

$$2(r-\lambda)(r^*-\lambda^*).$$

It is thus easily seen that the value of

$$\pi = \sum_{i=1}^{B} n_{ij}\, n_{i}m,\ j \neq m,$$

the total of all products turns out to be

$$\pi = 4\lambda\lambda^* + \lambda(b^* - 2r^* + \lambda^*) + 4\lambda(r^* - \lambda^*) + (b - 2r + \lambda)\lambda^* + 4(r - \lambda)\lambda^* + 2(r - \lambda)(r^* - \lambda^*).$$

Hence the theorem.

This result allows us to obtain balanced ternary design with the help of any two BIB designs in same number of factors. Following results follow from theorm 2.1.

**Theorem 2.2**

For all values of $v$ and $k$, the sets obtained by adding the elements of all the pairs $(i, j)$, $i \leqslant j$ of rows of $N_2$ form the blocks of a balanced ternary design with parameters

$$V = v,\ K = 2k,\ R = r(b+1),\ B = b + (b_2),\ \pi = (b+2)\lambda + r^2$$

The proof of this theorem also follows from theorem 2.1 by considering $N^*_2$ same as $N_2$ and by adding back $2r$, $b$ and $4\lambda$ in $R$, $B$ and $\pi$ respectively.

It may be observed that if from the ternary design, blocks obtained from all pairs $(i, i)$ are deleted along with the blocks obtained from pairs $(i', j')$ where the elements of blocks $i'$ and $j'$ are complementary to each other then also the remaining submatrix is a balanced ternary design. If we consider $N_2$ as the incidence matrix of an affine $\alpha$-resolvable BIB design, then by deleting the blocks in the manner explained above, we get balanced ternary designs obtained by Dey (1970). Thus, his series of ternary designs are straight away derivable from our series of designs. It may be seen that in the ternary design so constructed some of the blocks are repeated. If we consider each such block only once then also the design is balanced ternary.

**Corollary 2.1**

If in a BIB design $v \geqslant 2k$, then sets formed by adding the elements of all the pairs $(i, j)$; $i < j$ of rows of the incidence matrix $N_2$ generate a balanced ternary design with parameters

$$V = v,\ K = 2k,\ R = r(b-1),\ B = (b_2),\ \pi = (b-2)\lambda + r^2.$$

**Corollary 2.2**

When $v<2k$, then we obtain a balanced binary design with parameters

$$V=v,\ K=2k,\ R=r(b-1),\ B=(b_2),\ \pi=(b-2)\lambda+r^2$$

having frequencies 1 and 2 if all the varieties occur at least once in each of such pairs, and a balanced ternary design is obtained with frequencies 0, 1 and 2 and with same parameters if all the varieties do not occur even in one of such pairs.

**Theorem 2.3**

A balanced $(p+s-1)$-ary design can be obtained by adding the elements of pairs $(i, j)$ of rows where rows $i$ and $j$ are chosen from different incidence matrices $N_p$ and $N_s$ of any $p$-ary and $s$-ary designs in same number of factors.

**Corollary 2.3**

If $N_t$ be the incidence matrix of any balance $t$-ary design, then a balanced $(2t-1)$-ary design is balanced by adding the elements of all pairs $(i, j)$ of its row.

**Example 2.1**

Let us consider the BIB design $v=4$, $k=2$, $r=3$, $b=6$, $\lambda=1$. The $N_2$ is given as

$$N_2=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Then sets obtained by adding elements of all pairs $(i, j)$, $i\leqslant j$ of rows give us the following design matrix $N_3$ in 21 blocks

$$N_3=\begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

which is obviously a balanced ternary design with parameters

$$V=v=4,\ K=2k=4,\ R=21,\ B=b+(b_2)=21,\ \pi=17.$$

It may be seen that the blocks with elements (1 1 1 1) are repeated. If we consider such blocks only once we still get a balanced ternary design in 19 blocks with $\pi=15$. Similarly, if we delete the pairs $(i, i)$, then the blocks having elements of the type (2200) are deleted and we get a balanced ternary design (Theorem 2.2), in 15 blocks with $\pi=13$. If we delete the pairs $(i, i)$ and the pairs having elements complementary to each other, then the blocks having elements of the tyye (2200) and (1111) are deleted. The resulting submatrix again forms a balanced ternary design in 12 blocks with $\pi=10$.

**Example 2.2**

Let us consider the following incidence matrix of a balanced ternary design in 3 varieties

$$N_3=\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

and the binary (BIB) design $v=b=3$, $r=k=2$, $\lambda=1$ with incidence matrix

$$N_2=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then applying theorem 2.3 we get a balanced 4-ary design in 18 blocks and 6 plots :

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 0 | 2 | 3 | 1 | 2 | 2 | 2 |
| 3 | 2 | 1 | 2 | 2 | 2 | 2 | 1 | 3 |
| 2 | 3 | 1 | 1 | 3 | 2 | 1 | 2 | 3 |
| 3 | 2 | 1 | 3 | 1 | 2 | 1 | 3 | 2 |
| 3 | 1 | 2 | 3 | 0 | 3 | 1 | 2 | 3 |
| 2 | 2 | 2 | 2 | 1 | 3 | 0 | 3 | 3 |

It may also be observed that some of the blocks are repeated. We can delete the excess blocks to form a balanced 4-ary design in 10 blocks.

### 2.2 *Method 2*

In this we attempt to generalise the pattern of Method 1, Balanced $n$-ary designs in $(b_2)$ blocks can also be constructed by forming blocks through the addition of elements of triplets $(i_1, i_2, i_3)$ or quadruplets $(i_1, i_2, i_3, i_4)$ or so on, elements of higher order $t$-plets of all rows, $(i_1 < i_2 < i_3 \ldots < i_t)$.

**Example 2.3**

Consider the incidence matrix $N_2$ of Example 2.1. By adding the elements of triplets of rows of $N_2$ we get a balanced 4-ary design in $\binom{b}{3}$ blocks with $\pi = 42$.

## 3. Some Remarks

The results given in method 1 can be used to provide a systemetic procedure for obtaining balanced $n$-ary designs from the incidence matrix of any $(n-1)$-ary design. We state the following theorems without proof:

**Theorem 3.1**

Blocks formed through addition of elements of pair of $(i, i)$ from the incidence matrix $N_2$ of any BIB design and the blocks of the incidence matrix $N_2$ together from a balanced ternary design in $2b$ blocks.

The matrix $N_3$ so derived, contains two submatrices $(N_3)_1$, and $(N_3)_2$ of the same order where $(N_3)$ has the rows formed through addition of pairs $(i, i)$ from $N_2$ and $(N_3)_2$ is $N_2$. We, now, have the following result:

**Theorem 3.2**

Blocks formed through addition of elements of pairs $(i, i)$ from $(N_3)_1$ and the blocks of $N_3$ form a balanced 4-ary design.

Incidence matrix $N_4$ of this 4-ary design is again made up of two submatrices, say, $(N_4)_1$ and $(N_4)_2$ such that $(N_4)_1$ has the rows formed through addition of pairs $(i, i)$ from $(N_3)_1$ and $(N_4)_2$ is $N_3$. As a consequence of these theorems we have the following general result:

**Theorem 3.3**

Blocks formed through addition of elements of pairs $(i, i)$ from the submatrix $(N_t)_1$ along with the blocks of $N_t$ together form a balanced $(t+1)$-ary design.

## 4. Doubly Balanced $n$-ary Designs

Consider a doubly BIB design $(v, k, r, b, \lambda, \mu)$ which has the additional property that any triplet of treatments occur exactly in $\mu$ blocks [cf Calvin (1954)].

We now define a doubly balanced $n$-ary design.

**Definition 1**

Any $n$-ary design is doubly balanced if

$$\sum_{j \neq m \neq s}^{B} n_{ij}\, n_{im}\, n_{is} = \text{constant} = \theta$$

This is obviously analogous to a doubly balanced incomplete block design.

It may now be observed that if any $n$-ary design is obtained by using a doubly BIB design by following the methods 1 or 2, then the $n$-ary design is also doubly balanced. For instance the balanced ternary design obtained in Example 2.1 is doubly balanced because it has been obtained through a doubly BIB design. When any $n$-ary design is obtained through theorem 2.1, then the design will be doubly balanced only if both the designs $N_2$ and $N^*_2$ are doubly balanced. This can easily be shown by extending the arguments of the proof of theorem 2.1.

## 5. Partially Balanced Arrays

If there exists a BIB design with parameters $(v, k, r, b, \lambda)$, then Chakravarti (1961) showed that the incidence matrix $N_2$ is a partially balanced array with $v$ constraints, $b$ assemblies, in two symbols 0 and 1 and of strength 2. Similarly the incidence matrix of the doubly BIB design $(v, k, r, b, \lambda, \mu)$ is a partially balanced array of strength 3. Some other results in this direction are also reported by Chakravarti (1961, 1963). We now state the following theorem :

**Theorem 5.1**

If the incidence matrix $N_2$ of any BIB design implies the existence of a partially balanced array with $v$ constraints, $b$ assemblies, two symbols 0 and 1 and of strength $t$, then the incidence matrix $N_3$ of the balanced ternary design (Theorems 2.1 and 2.2) is a partially balanced array of strength $t$ with $\binom{b}{2}$ or $b+\binom{b}{2}$ assemblies, $v$ constraints and is in three symbols 0, 1 and 2.

The proof of the theorem is obvious. Similarly, in general the balanced $n$-ary designs have the incidence matrices which are partially balanced arrays of strength $t$ in $n$ symboles 0, 1, 2, ..., $n-1$. It can also be seen that if a balanced ternary design is obtained through theorem 2.1, then by extending the arguments of the proof of theorem 2.1 and section 4 it can be shown that incidence matrix $N_3$ of this ternary design is a partially balanced array of strength $t$ where $t$ min $(t_1, t_2)$, $t_1$ and $t_2$ being the strengths of partially balanced array of $N_2$ and $N^*_2$. For instance, if $N_2$ is the incidence matrix of BIB design such that it is partially balanced array of strength 2 and $N^*_2$ is the incidence martix of doubly BIB design so that it is a partially balanced array of strength 3, then $N_3$, the incidence martix of the ternary design is only a balanced design (Section 4) and is a partially balanced array of strength 2. It may be seen that the design martix $N_3$ of example 2.1 is a partially balanced array of strength 4 $(=v)$


## Summary

In this paper we have evolved some methods for constructing Balanced $n$-ary Block Designs by using the incidence martix of a BIB Design. We have shown that balanced $n$-ary design can be obtained through any two $p$-ary and $s$-ary balanced block designs. It is then shown that the incidence matrices of these designs can be considered as partially balanced array of strength $t$.


## References


Chakravarti, I.M. (1961) On some methods of constructions of partially balanced arrays. Ann. Math. Statist. 33, 1181-1185.

Chakravarti, I.M. (1963) : Orthogonal and partially balanced arrays—their applications in design of experiments. Metrika, 7, 231-243.

Dey, A. (1970) : On Construction of Balanced $n$-ary block designs. Ann. Instt. Statist. Math. 22, 389-393.

Rao, S.V.S.P. (1966) : On construction of Balanced $n$-ary designs. Diploma Thesis, I.A.R.S., New Delhi.

Tocher, K.D. (1952) : Design and Analysis of block experiments. Jour. Roy. Statist. Soc. B, 14, 45-100.